MULTIPLE-SERVERS QUEUE WITH BULK ARRIVALS

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ABSTRACT

The transient and steady-state behaviour of the M/M/r queueing process with bulk arrivals is analysed. The transient behaviour is treated in terms of Laplace transforms, and steady-state behaviour - in terms of generating functions and probabilities. The influence of the bulk-size variance on the expected queue size is discussed at some length.

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INTRODUCTION

In many queueing situations, customers arrive in bulks and are served individually. Such is the case, for example, in an airport, where passengers arrive as a single group in a plane but are served individually at the passport-control and customes counters. The present paper deals with a system where bulks with randomly distributed size arrive, in a stationary Poisson stream, at a single queue attended by r servers, the queue discipline being first-come-first-served. Members of a newly-arrived bulk join the end of the queue in a random order. Service times are assumed to be mutually independent, with an identical negative exponential distribution. The transient and steady-state behaviour of the system is analysed, and some closed-form results concerning queue sizes are derived.

A similar bulk-arrivals model, in which service is provided in batches, was derived by Loris-Tegham [1] who formulated the Kolmogorov equations and obtained solutions for special cases. The case $r = 1$, a single-server system, was studied extensively by Gaver [2] using the imbedded Markov chain method.

Here the solution is approached through Kolmogorov's forward differential-difference equations, using generating functions and their Laplace transforms. In the steady-state situation, particular attention is given to the effect of the mean and distribution of the bulk size on the average queue size.

MATHEMATICAL MODEL

a. Transient behaviour

The bulk arrival rate is denoted by λ and bulk size is assumed to be arbitrarily distributed as a random variable N, possessing a probability mass function $f_N(n)$, $n=1,2,\ldots$. The service rate for each of the r servers is denoted by μ.

Let $P_{ji}(t)$ denote the probability of there being i customers in the system at time t, given that at $t=0$ their number equals j. For simplicity, we omit the subscript j and use $P_i(t)$ instead of $P_{ji}(t)$. Kolmogorov's forward differential-difference equations take the following form:

$$\frac{d}{dt} P_i(t) = \lambda \sum_{m=0}^{i-1} f_N(i-m)P_m(t) + \mathrm{Min}\{i+1,r\}\mu P_{i+1}(t) - (\lambda+\min\{i,r\}\mu)P_i(t) , \qquad i=1,2,\ldots , \tag{1}$$

and

$$\frac{d}{dt} P_o(t) = - \lambda P_o(t) + \mu P_1(t) . \tag{2}$$

Note that $f_N(n) = 0$ for $n<1$.

We now define two generating functions:

$$G(z,t) = \sum_{i=0}^{\infty} z^i P_i(t) , \qquad |z| \leq 1 , \tag{3}$$

$$G_N(z) = \sum_{n=0}^{\infty} z^n f_N(n) , \qquad |z| \leq 1 . \tag{4}$$

The Laplace transforms of $G(z,t)$ and $P_i(t)$ are defined as

$$G^*(z,s) = \int_0^{\infty} e^{-st} dG(z,t) , \qquad \mathrm{Re}(s)>0 , \tag{5}$$

and

$$P_i^*(s) = \int_0^{\infty} e^{-st} dP_i(t) , \qquad \mathrm{Re}(s)>0 . \tag{6}$$

The Laplace transform $G^*(z,s)$ is obtained from Eqs. (1) and (2) by taking first the generating functions and then referring to relations (5) and (6):

$$G^*(z,s) = \frac{z^{j+1}+(z-1)\mu_i\sum_{i=0}^{r-1}(r-i)z^iP_i^*(s)}{sz+\lambda z(1-G_N(z))+(z-1)r\mu}, \tag{7}$$

where j is the number in the system at time $t=0$.

In Eq. (7) the Laplace transforms $P_i^*(s)$, $i=0,1\ldots,r-1$ are unknown. The following method is suggested for determining these functions: define a generating function

$$Q(z,t) = \sum_{i=0}^{r-2} z^iP_i(t), \qquad |z| \leq 1 . \tag{8}$$

From Eqs. (1) and (2), we obtain

$$\frac{\partial}{\partial t} Q(z,t) - \mu(1-z)\frac{\partial}{\partial z} Q(z,t) =$$

$$\lambda\sum_{i=1}^{r-2} z^i \sum_{m=0}^{i-1} f_N(i-m)P_m(t) - \lambda Q(z,t)+(r-1)\mu z^{r-2}P_{r-1}(t) . \tag{9}$$

This nonhomogeneous partial linear differential equation can be solved for $Q(z,t)$ in terms of $P_i(t)$, $i=0,1,\ldots,r-2$. The solution is obtained in the form $\mathcal{F}(u(z,t,Q), v(z,t,Q)) = 0$, where

$$\text{and}\quad \begin{aligned} u(z,t,Q) &= C_1, & C_1 &= \text{constant} ,\\ v(z,t,Q) &= C_2, & C_2 &= \text{constant} ,\end{aligned}$$

satisfy the equations

$$\frac{dt}{1} = \frac{dz}{\mu(z-1)} = \frac{-\,dQ(z,t)}{\lambda Q(z,t)-\lambda\sum_{i=1}^{r-2} z^i \sum_{m=0}^{i-1} f_N(i-m)P_m(t)-(r-1)\mu z^{r-2}P_{r-1}(t)} . \tag{10}$$

The functional form of $\mathcal{F}$ is obtainable from the initial condition of the process (For details see Sneddon [3]; the same method is used by Saaty [4] in solving the M/M/r process).

After some rather lengthy manipulations, the desired solution is obtained in the following form:

$$Q(z,t) = \sum_{i=1}^{r-2} \sum_{m=0}^{i-1} \int_0^t e^{\lambda(t-u)} (1-(1-z)e^{-\mu(t-u)})^i f_N(i-m) P_m(u)du$$

$$+ Q(z,0) + (r-1) \int_0^t e^{\lambda(t-u)} (1-(1-z)e^{-\mu(t-u)})^{r-2} P_{r-1}(u)du \tag{11}$$

where

$$Q(z,0) = \begin{cases} 0 & \text{if} \quad j > r-2 \\ z^j & \text{otherwise} . \end{cases}$$

Taking the Laplace transform of Eq. (11), we have (note that the right-hand side of Eq. (11) is given in forms of convolutions):

$$\sum_{i=0}^{r-2} z^i \, P_i^*(s) = \sum_{i=1}^{r-2} \sum_{m=0}^{i-1} \sum_{w=0}^{i} \frac{f_N(i-m) \binom{i}{w} (z-1)^{i-w}}{s+(i-w)\mu+\lambda} P_m^*(s)$$

$$+ Q(z,0) + (r-1) \sum_{w=0}^{r-2} \frac{\binom{r-2}{w} (z-1)^{r-2-w}}{s+(r-2-w)\mu+\lambda} P_{r-1}^*(s) \quad . \tag{12}$$

Equation (12) is an identity of two polynomials of degree $r-2$ in z. The coefficients of equal powers of z on both sides must be identical, hence, $(r-1)$ linear equations are obtainable, relating $P_i^*(s)$, $i=0,1,\ldots,r-1$; these are conveniently derived by repeated differentiation with respect to z at $z = 0$, and yield

$$P_i^*(s) = \frac{\lambda}{\lambda+s} \sum_{w=0}^{i-1} f_N(i-w) P_w^*(s) + \sum_{n=i}^{r-2} \sum_{w=0}^{n-1} \sum_{\xi=0}^{n-i} (-1)^{\xi} \frac{\lambda f_N(n-w) \binom{n}{i} \binom{n-i}{\xi}}{\lambda + \mu\xi + s} P_w^*(s)$$

$$+ \mu P_{r-1}^*(s)(r-1) \binom{r-2}{i} \sum_{\xi=0}^{r-i-2} (-1)^{\xi} \frac{\binom{r-i-2}{\xi}}{\lambda+\xi\mu+s} + \Psi , \qquad i=0,1,\ldots,r-2, \tag{13}$$

where

$$\Psi = \begin{cases} 1 & \text{if} \;\; i = j \\ 0 & \text{otherwise.} \end{cases}$$

Note that in Eq. (13), and throughout this paper, a sum in which the upper limit is less than the lower limit is defined as equal to zero.

We now have $r-1$ linear equations and r unknowns, namely $P_i^*(s)$, $i=0,1,\ldots,r-1$. The missing relation is obtained from Eq. (7). $G^*(z,s)$ is analytic with respect to z in $|z| \le 1$. The denominator of Eq. (7) has exactly one zero in $|z| \le 1$, which must also be a zero of the numerator yielding the missing linear equation in $P_i^*(s)$, $i=0,1,\ldots,r-1$. To prove that our denominator has

one zero as above, let

$$g(z) = -\lambda z G_N(z) - r\mu ,$$

and

$$f(z) = (\lambda + r\mu + s)z .$$

For $|z| = 1$ we have

$$|f(z)| > |g(z)| .$$

It follows then from Rouche's Theorem that $f(z) + g(z)$ (which is our denominator) and $f(z)$ have the same number of zeros in $z < 1$, and since $f(z)$ has exactly one zero in $|z| < 1$ (at $z=0$), the denominator of (7) has exactly one zero in $|z| < 1$. On $|z| = 1$ the denominator cannot equal zero, since

$$|f(z) + g(z)| \geq |f(z)| - |g(z)| > 0 ,$$

and the proof is complete.

The single zero in question is real for $s = \mathrm{Re}(s)$. If we denote it by z_o, the additional linear equation takes the form

$$(z_o - 1)\ \mu \sum_{i=0}^{r-1} (r-i) z_o^i\, P_i^*(s) + z_o^{j+1} = 0 . \tag{14}$$

Clearly, only numerical solutions are possible for the general case, since z_o is unobtainable in closed form. For bulk sizes not exceeding 3, z_o can be determined in closed form as a solution of a polynomial whose degree is the maximum bulk size plus one.

The form of $P_i^*(s)$, $i \geq r$, is readily obtained from Eq. (7)

$$P_i^*(s) = \mu \sum_{n=0}^{r-1} (r-n) P_n^*(s) \left[\frac{A(z,i-n-1)}{(i-n-1)!} - \frac{A(z,i-n)}{(i-n)!} \right] \Bigg|_{z=0} + \frac{A(z,i-j-1)}{(i-j-1)!}\, \eta \Bigg|_{z=0} , \tag{15}$$

where $\eta = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise,} \end{cases}$

$$A(z,w) = \frac{\partial^w}{\partial z^w} \left(sz + \lambda z(1-G_N(z)) + (z-1)r\mu\right)^{-1} ,$$

and $A(z,w) = 0$ if $w<0$.

b. Steady-state behaviour

The steady-state generating function for the number in the system, denoted by $G(z)$, is obtained as follows:

$$G(z) = \lim_{s\to 0} sG^*(z,s) = \frac{(1-z)\mu\sum_{\xi=0}^{r-1}(r-i)z^i P_i}{\lambda z(G_N(z)-1)+(1-z)r\mu}, \tag{16}$$

where $P_i = \lim_{t\to\infty} P_i(t)$.

The values of P_i, $i=0,1,\ldots,r-1$, are obtainable as the solution of the following set of r linear equations:

$$P_i = \sum_{w=0}^{i-1} f_N(i-w)P_w + \sum_{n=i}^{r-2}\sum_{w=0}^{n-1}\sum_{\xi=0}^{n-i}(-1)^\xi \frac{\lambda f_N(n-w)\binom{n}{i}\binom{n-i}{\xi}}{\lambda+\mu\xi}$$

$$+ (r-1)\mu P_{r-1}\binom{r-2}{i}\sum_{\xi=0}^{r-i-2}(-1)^\xi \frac{\binom{r-2-i}{\xi}}{r+\mu\xi}, \qquad i=0,1,2,\ldots,r-2, \tag{17}$$

and

$$\mu\sum_{i=0}^{r-2}(r-1)P_i = r\mu - \lambda E(N). \tag{18}$$

The $(r-1)$ equations given by (17) were obtained as the limit of equations (13), and the r-th equation, (18), is obtained by setting $z=1$ in equation (16). It can be shown that the steady-state condition is

$$\rho = \frac{\lambda E(N)}{r\mu} < 1 . \tag{19}$$

For example, for $r = 3$, the solution of (17) and (18) is:

$$P_0 = \frac{3 - r\rho}{3 + 2\frac{\lambda}{\mu} + \frac{\lambda}{2\mu}\left(\frac{\lambda}{\mu} + 1 - f_N(1)\right)},$$

$$P_1 = \frac{\lambda}{\mu} P_0, \qquad \text{and} \qquad P_2 = \frac{1}{2}\left(\left(1+\frac{\lambda}{\mu}\right)\frac{\lambda}{\mu} - \frac{\lambda}{\mu} f_N(1)\right)P_0 . \tag{20}$$

The expected number of customers in the system, denoted by L, is readily obtained from Eq. (16) as

$$L = \frac{1}{2r(1-\rho)^2} \sum_{i=0}^{r-1}\left(\left(\frac{E(N)^2}{E(N)} + 1\right)\rho + 2i(1-\rho)\right)(r-i)P_i , \tag{21}$$

and the variance of the number in the system, denoted by V, is obtained as:

$$V = \sum_{i=0}^{r-1} \left(\frac{3\rho(\frac{E(N^2)}{E(N)} + 1)i + \rho(\frac{E(N^3)}{E(N)} - 1) + 3i(i-1)(1-\rho)}{3r(1-\rho)^2} + \frac{\rho^2(\frac{E(N^2)}{E(N)} + 1)^2}{2r(1-\rho)^3} \right) \cdot (r-i)P_i + L(1-L) \quad . \tag{22}$$

In the special case where N is a constant, say $N = C$, P_i, $i=0,1,\ldots,r-1$, is obtainable in closed form as follows:

$$P_0 = \frac{r(1-\rho)}{r + \sum_{n=1}^{r-1} \left(\frac{r-n}{n!} \prod_{m=0}^{n-1} \left(\frac{\lambda}{\mu} + m \right) \right)} , \tag{23}$$

and

$$P_i = \frac{P_0}{i!} \prod_{m=0}^{i-1} \left(\frac{\lambda}{\mu} + m \right) , \qquad i=1,2,\ldots,r-1 .$$

Substitution in (21) and (22) yields closed-form expressions for L and V.

Figure 1 gives the value of L/r as function of the expected bulk size, $E(N)$, for different values of r with ρ kept constant at 0.9. The solid lines represent the case of constant bulk size and the dashed lines that of geometrically-distributed bulk size. The figure shows that L increases approximately linearly with $E(N)$. Also, for the geometrical distribution L is greater compared to the constant bulk size case.

Intuitively, one would expect L to be smallest when the bulk size is constant, that is $f_N(E(N)) = 1$. This conjecture is also supported by Figure 1, where L for the constant bulk size was found to be smaller than for its geometrically distributed counterpart with identical mean. Here, only one sufficient condition will be proved, a more general proof being too complicated.

Theorem: Let A and B be two queueing systems of the type described in this paper, identical in all respects except for the bulk-size distribution. If the bulk sizes in A and B possess probability mass functions $f_{N_A}(.)$ and $f_{N_B}(.)$ respectively, such that $E(N_A) = E(N_B) = C \geq r-1$, and $f_{N_A}(i) = 0$ for $i=0,1,\ldots,r-2$, then $L_A > L_B$ if and only if $V(N_A) > V(N_B)$.

Proof: Let $P_i(A)$ and $P_i(B)$, $i=0,1,2,\ldots$ denote the steady-state probabilities of there being i customers in system A and B, respectively. From Eqs. (17) and (18) it follows that:

$$P_i(A) = P_i(B) , \qquad i=0,1,\ldots,r-1 . \tag{24}$$

Substitution of Eq. (24) in (21) yields:

$$L_B - L_A = \frac{V(N_B) - V(N_A)}{2rC(1-\rho)^2} \sum_{i=0}^{r-1} (r-i)P_i(A) \quad . \tag{25}$$

From Eq. (25) it is seen that $(L_B - L_A)$ is an increasing linear function of $(V(N_B) - V(N_A))$. For the special case where the bulk size in system A is constant, we have:

$$L_B - L_A = \frac{V(N_B)}{2rC(1-\rho)^2} \sum_{i=0}^{r-1} (r-i)P_i(A) \geq 0 \quad . \tag{26}$$

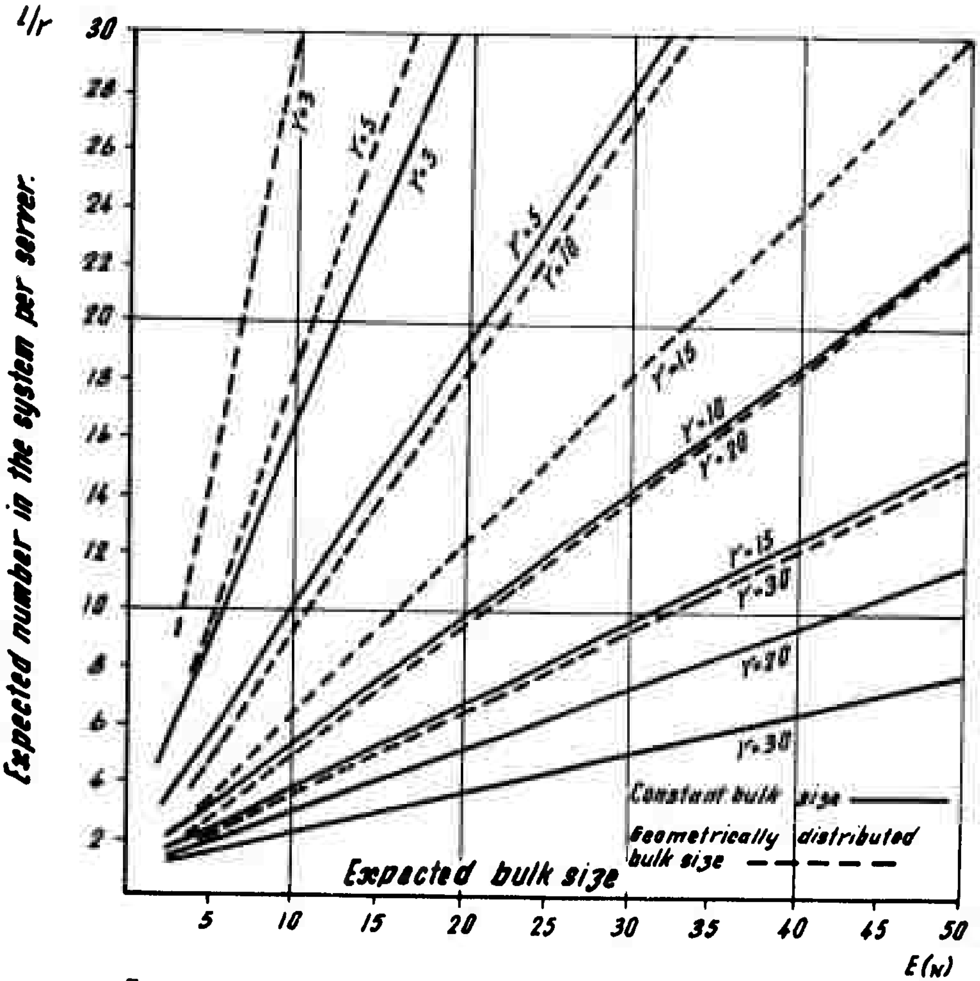

Figure 1: *Expected number of customers in the system per server as a function of the expected bulk size, ($\rho = 0.9$).*

R E F E R E N C E S

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